Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Asymptotics beats Monte Carlo: The case of correlated local vol baskets

Christian Bayer and Peter Laurence

WIAS Berlin and Università di Roma

$$u(t, S_t) = e^{-r(T-t)} E\left[f(S_T) \mid S_t \right]$$

Example (Example treated in this work)

- $f(\mathbf{S}) = \left(\sum_{i=1}^{n} w_i S_i - K\right)^+$, at least one weight positive
- n large (e.g., $n = 500$ for SPX)

- PDE methods
- (Quasi) Monte Carlo method
- Fourier transform based methods
- Approximation formulas

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- PDE methods

  Pros: fast, general

  Cons: curse of dimensionality, path-dependence may or may not be easy to include
- (Quasi) Monte Carlo method
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- n large (e.g., $n = 500$ for SPX)

- PDE methods
- (Quasi) Monte Carlo method

  Pros: very general, easy to adapt, no curse of dimensionality

  Cons: slow, quasi MC may be difficult in high dimensions
- Fourier transform based methods
- Approximation formulas

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  Pros: very fast to evaluate ("explicit formula")

  Cons: only available for affine models, difficult to generalize, curse of dimensionality
- Approximation formulas

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Pros: very fast evaluation

Cons: derived on case by case basis, therefore very restrictive

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- Fourier transform based methods
- Approximation formulas
- Work horse methods: PDE methods and (in particular) (Q)MC
- Particular models allowing approximation formulas (e.g., *SABR formula*) or FFT (Heston model) very popular

Outline

Setting

- Local volatility model for forward prices

$$dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \ldots, n,$$
$$\left\langle dW_i(t)\,,\, dW_j(t) \right\rangle = \rho_{ij}dt$$

- Generalized spread option with payoff $\left(\sum_{i=1}^{n} w_i F_i - K\right)^+$, at least one w_i positive
- Goal: fast and accurate approximation formulas, even for high n
- $n = 100$ or $n = 500$ not uncommon (index options)

Example

- Black-Scholes model: $\sigma_i(F_i) = \sigma_i F_i$
- CEV model: $\sigma_i(F_i) = \sigma_i F_i^{\beta_i}$

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- Consider the basket (index) $\sum_{i=1}^{n} w_i F_i$:

$$d \sum_{i=1}^{n} w_i F_i(t) = \sum_{i=1}^{n} w_i \sigma_i(F_i(t)) dW_i(t)$$

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$$\left(\sum_{i=1}^{n} w_i F_i(t) - K\right)^+ = \left(\sum_{i=1}^{n} w_i F_i(0) - K\right)^+ +$$
$$+ \sum_{i=1}^{n} w_i \int_0^T \mathbf{1}_{\sum w_i F_i(u) > K} dF_i(u) + \frac{1}{2}\int_0^T \delta_{\sum w_i F_i(u) = K} \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}(u)) du$$

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- Ito's formula formally implies with $\mathcal{E}(K) = \{\mathbf{F} | \sum w_i F_i = K\}$ that

$$C(\mathbf{F}(0), K, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K \right)^{+} + \frac{1}{2} \int_0^T E\left[\sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}(u)) \delta_{\mathcal{E}(K)}(\mathbf{F}(u)) \right] du$$

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$$C(\mathbf{F}(0), K, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K \right)^+ + \frac{1}{2\,|\mathbf{w}|} \int_0^T \int_{\mathbb{R}^n} |\nabla v(\mathbf{F})|\, \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}) \delta_0(v(\mathbf{F}) - K) p(\mathbf{F}_0, \mathbf{F}, u) d\mathbf{F} du$$

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- Let H_{n-1} be the Hausdorff measure on $\mathcal{E}(K)$. Recall the co-area formula:

$$\int_\Omega |\nabla v(x)| g(x) dx = \int_{-\infty}^{\infty} \int_{v^{-1}(\{s\})} g(x) H_{n-1}(dx) ds$$

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- Let H_{n-1} be the Hausdorff measure on $\mathcal{E}(K)$, then we have the *Carr-Jarrow formula*

$$C_{\mathcal{B}}(\mathbf{F}_0, K, T) = \left(\sum_{i=1}^n w_i F_i(0) - K\right)^+ +$$
$$+ \frac{1}{2|\mathbf{w}|} \int_0^T \int_{-\infty}^{\infty} \delta_0(s - K) \int_{\mathcal{E}_s} \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}) p(\mathbf{F}_0, \mathbf{F}, t) H_{n-1}(d\mathbf{F}) ds dt$$

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$$+ \frac{1}{2} \int_0^T \frac{1}{|w|} \int_{\mathcal{E}(K)} \sum_{i,j=1}^{n} w_i w_j \sigma_i(F_i) \sigma_j(F_j) \rho_{ij} p(\mathbf{F}_0, \mathbf{F}, u) H_{n-1}(d\mathbf{F}) du.$$

- Heat kernel expansion (to be discussed in detail later):

$$\sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F})p(\mathbf{F}_0,\mathbf{F},t) \approx \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{d(\mathbf{F}_0,\mathbf{F})^2}{2t} - C(\mathbf{F}_0,\mathbf{F})\right)$$

- By change of variables $F_n = \frac{1}{w_n}\left(K - \sum_{i=1}^{n-1} w_i F_i\right)$ on $\mathcal{E}_K$:

$$H_{n-1}(d\mathbf{F}) = \frac{|w|}{|w_n|} dF_1 \cdots dF_{n-1}$$

- Laplace approximation: with $\mathbf{F}^* = \operatorname{argmin}_{\mathbf{F}\in\mathcal{E}_K} d(\mathbf{F}_0,\mathbf{F})$ and $\mathcal{G}_K = \{(F_1,\ldots,F_{n-1}) | \sum_{i=1}^{n-1} w_i F_i < K\}$

$$\int_{\mathcal{G}_K} e^{-\frac{d(\mathbf{F}_0,\mathbf{F})^2}{2t} - C(\mathbf{F}_0,\mathbf{F})} dF_1 \cdots dF_{n-1} \approx e^{-\frac{d(\mathbf{F}_0,\mathbf{F}^*)^2}{2t} - C(\mathbf{F}_0,\mathbf{F}^*)} \int_{\mathbb{R}^{n-1}} e^{-\frac{\mathbf{z}^T Q \mathbf{z}}{2t}} d\mathbf{z}$$

$$= t^{\frac{n-1}{2}} e^{-\frac{d(\mathbf{F}_0,\mathbf{F}^*)^2}{2t} - C(\mathbf{F}_0,\mathbf{F}^*)} \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{\det Q}}$$

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Non-degeneracy of the optimization problem

- Assume non-degeneracy of $\mathbf{F}^* = \mathrm{argmin}_{\mathbf{F}\in\mathcal{E}_K} d(\mathbf{F}_0, \mathbf{F})$
- Generically true, but exceptional points $\mathbf{F}_0$ or K often exist.
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- Example: F_1, F_2 independent, Black-Scholes assets, $\sigma_i = 1$, $F_{0,i} = 1$, f ... density of $F_{1,T} + F_{2,T}$. Then

$$f(K) = \begin{cases} \exp\left(-\frac{\Lambda(K)}{T}\right) \frac{1}{\sqrt{T}} \left(c_0 + O(T)\right), & K \neq 2e, \\ \exp\left(-\frac{\Lambda(K^*)}{T}\right) \frac{1}{T^{3/4}} \left(c_0 + O(T)\right), & K = 2e. \end{cases}$$

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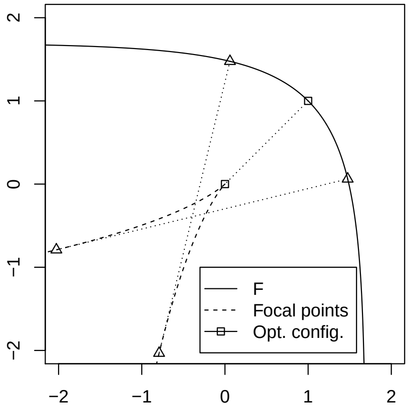
- Related concept of focality in Riemannian geometry.



Theorem

$$C_{\mathcal{B}}(\mathbf{F}_0, K, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K \right)^+ +$$

$$+ \frac{1}{2\sqrt{2\pi}\,|w_n|\,d(\mathbf{F}_0, \mathbf{F}^*)^2 \sqrt{\det Q}} e^{-C(\mathbf{F}_0, \mathbf{F}^*) - \frac{d(\mathbf{F}_0, \mathbf{F}^*)}{2T}} T^{3/2} + o(T^{3/2}), \text{ as } T \to 0.$$

- Bachelier implied vol (with $\overline{F}_0 = \sum_{i=1}^{n} w_i F_{0,i}$):

$$\sigma_B \sim \sigma_{B,0} + T\sigma_{B,1} \text{ with } \sigma_{B,0} = \frac{\left|\overline{F}_0 - K\right|}{d(\mathbf{F}_0, \mathbf{F}^*)\left|\overline{F}_0\right|}, \ \sigma_{B,1} = \cdots$$

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The ATM case

- Above formulas have singularities when $\overline{F}_0 = K$ (ATM)
- Resolve by *l'Hopital formula* or *first order* heat kernel expansion.
- We have $\mathbf{F}^* = \mathbf{F}_0$ and

$$\det Q = \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0) \det \rho^{-1} \prod_{k=1}^{n} \sigma_k(F_{0,k})^{-2} / w_n^2.$$

- Higher order Laplace exansion required.
- $\sigma_{BS,0} = \sigma_{Bach,0} = \dfrac{\sigma_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0)}{\overline{F}_0}$
- $\sigma_{BS,1} = \dfrac{\sqrt{2\pi}}{3K}\left(g_1^{\bar{u}_0} + g_0^{\bar{u}_1}\right) + \dfrac{\sigma^3_{BS,0}}{24} = \sigma_{Bach,1} + \dfrac{\sigma^3_{BS,0}}{24}$

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- Goal: sensitivity w. r. t. model parameter κ of the option price

$$C_{\mathcal{B}}(\mathbf{F}_0, K, T) \approx C_{BS}(\overline{F}_0, K, \sigma_{BS}, T)$$

- Sensitivity: $\underbrace{\partial_\kappa C_{BS}}_{\text{BS greek}}(\overline{F}_0, K, \sigma_{BS}, T) + \underbrace{\nu_{BS}}_{\text{BS vega}}(\overline{F}_0, K, \sigma_{BS}, T)\partial_\kappa \sigma_{BS}$
- Recall that $\sigma_{BS,0}, \sigma_{BS,1}$ explicit up to $\mathbf{F}^*$
- By the minimizing property: $\partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\Big|_{\mathbf{G}=\mathbf{G}^*} = 0$
- Differentiating with respect to κ gives

$$\partial_\kappa \partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\Big|_{\mathbf{G}^*} + \sum_{l=1}^{n-1} \partial_{F_l}\partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\Big|_{\mathbf{G}^*} \partial_\kappa F_l^* = 0$$

Up to the above system of linear equations for $\partial_\kappa \mathbf{F}^*$, there are explicit expression for the sensitivities of the approximate option prices.

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- Sensitivity: $\partial_\kappa C_{BS}(\overline{F}_0, K, \sigma_{BS}, T) + \nu_{BS}(\overline{F}_0, K, \sigma_{BS}, T)\partial_\kappa \sigma_{BS}$
- Recall that $\sigma_{BS,0}, \sigma_{BS,1}$ explicit up to $\mathbf{F}^*$
- By the minimizing property: $\partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\Big|_{\mathbf{G}=\mathbf{G}^*} = 0$
- Differentiating with respect to κ gives

$$\partial_\kappa \partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\Big|_{\mathbf{G}^*} + \sum_{l=1}^{n-1} \partial_{F_l}\partial_{F_i} d^2\left(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G})\right)\Big|_{\mathbf{G}^*} \partial_\kappa F_l^* = 0$$

Up to the above system of linear equations for $\partial_\kappa \mathbf{F}^*$, there are explicit expression for the sensitivities of the approximate option prices.

Outline

Heat kernels and geometry

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

$$L = \frac{1}{2}a^{i,j}\frac{\partial^2}{\partial x^i \partial x^j} + b^i \frac{\partial}{\partial x^i}, \quad a = \sigma^T \sigma$$

- Heat kernel: fundamental solution $p(\mathbf{x}, \mathbf{y}, t)$ of $\frac{\partial}{\partial t}u = Lu$
- Transition density of $\mathbf{X}_t$

"Can you hear the shape of the drum?"(Kac '66)

Take $L = \Delta$ on a domain D and relate:

- Geometrical properties of the domain D
- Partition function $Z = \sum_{k \in \mathbb{N}} e^{\gamma_k t}$
- Heat kernel
- E.g. $-\gamma_k \sim C(n)(k/\operatorname{vol} D)^{2/n}$ (Weyl, '46)
- E.g. (for $n = 2$): $Z = \frac{\text{area}}{4\pi t} - \frac{\text{circ.}}{\sqrt{4\pi t}} + O(1)$ (McKean & Singer, '67)

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The Riemannian metric associated to a diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

$$L = \frac{1}{2}a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + b^i \frac{\partial}{\partial x^i}, \quad a = \sigma^T \sigma$$

- On $\mathbb{R}^n$ (or a submanifold), introduce $g^{ij} := a^{ij}$, Riemannian metric tensor $(g_{ij}(\mathbf{x}))_{i,j=1}^n := \left((g^{ij}(\mathbf{x}))_{i,j=1}^n\right)^{-1}$
- Geodesic distance:

$$d(\mathbf{x},\mathbf{y}) := \inf_{\mathbf{z}(0)=\mathbf{x},\,\mathbf{z}(1)=\mathbf{y}} \int_0^1 \sqrt{\sum g_{ij}(\mathbf{z}(t))\dot{\mathbf{z}}^i(t)\dot{\mathbf{z}}^j(t)}dt$$

- inf attained by a smooth curve, the *geodesic*
- Laplace-Beltrami operator: $\Delta_g = \left(\det(g_{ij})\right)^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left(\det(g_{ij})\right)^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x^j}$

$$L = \frac{1}{2}a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + b^i \frac{\partial}{\partial x^i} = \frac{1}{2}\Delta_g + h^i \frac{\partial}{\partial x^i}$$

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Heat kernel expansion

$$p_N(\mathbf{x}_0, \mathbf{x}, T) = \sqrt{\det(g(\mathbf{x})_{ij})} U_N(\mathbf{x}_0, \mathbf{x}, T) \frac{e^{-\frac{d^2(\mathbf{x}_0, \mathbf{x})}{2T}}}{(2\pi T)^{\frac{n}{2}}}$$

- $U_N(\mathbf{x}_0, \mathbf{x}, T) = \sum_{k=0}^{N} u_k(\mathbf{x}_0, \mathbf{x}) T^k$, the heat kernel coefficients
- $u_0(\mathbf{x}_0, \mathbf{x}) = \sqrt{\Delta(\mathbf{x}_0, \mathbf{x})} e^{\int_z \langle h(z(t)), \dot{z}(t) \rangle_g dt}$
- Δ is the Van Vleck-DeWitt determinant:
  $\Delta(\mathbf{x}_0, \mathbf{x}) = \frac{1}{\sqrt{\det(g(\mathbf{x}_0)_{ij}) \det(g(\mathbf{x})_{ij})}} \det\left(-\frac{1}{2} \frac{\partial^2 d^2}{\partial \mathbf{x}_0 \partial \mathbf{x}}\right)$.
- $e^{\int_z \langle h(z(t)), \dot{z}(t) \rangle_g dt}$ is the exponential of the work done by the vector field h along the geodesic z joining $\mathbf{x}_0$ to $\mathbf{x}$ with
  $h^i = b^i - \frac{1}{2\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^j} \left[\sqrt{\det(g_{ij})} g^{ij} \right]$

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Assumption

The cut-locus of any point is empty,

Theorem (Varadhan '67)

$b = 0$, σ *uniformly Hölder continuous, system uniformly elliptic, then* $\lim_{T\to 0} T \log p(\mathbf{x}, \mathbf{y}, T) = -\frac{1}{2} d(\mathbf{x}, \mathbf{y})^2$.

Theorem (Yosida '53)

On a compact Riemannian manifold, assume smooth vector fields and an ellipticity property. Then $p(\mathbf{x}, \mathbf{y}, T) - p_N(\mathbf{x}, \mathbf{y}, T) = O(T^N)$ *as* $T \to 0$.

Theorem (Azencott '84)

For a locally elliptic system in an open set $U \subset \mathbb{R}^n$, $\mathbf{x}, \mathbf{y} \in U$ *s. t.* $d(\mathbf{x}, \mathbf{y}) < d(\mathbf{x}, \partial U) + d(\mathbf{y}, \partial U)$, *we have* $p(\mathbf{x}, \mathbf{y}, T) - p_N(\mathbf{x}, \mathbf{y}, T) = O(T^N)$ *as* $T \to 0$.

The local vol case

- Domain $\mathbb{R}_+^n$, $dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \ldots, n$
- $L = \frac{1}{2}\rho_{ij}\sigma_i(x^i)\sigma_j(x^j)\frac{\partial^2}{\partial x^i \partial x^j}$
- Let $A \in \mathbb{R}^{n\times n}$ be such that $A\rho A^T = I_n$. Change variables $\mathbf{F} \to \mathbf{y} \to \mathbf{x}$ according to

$$y_i = \int_0^{F_i} \frac{du}{\sigma_i(u)}, \; i = 1, \ldots, n, \quad \mathbf{x} = A\mathbf{y}, \quad L \to \frac{1}{2}\frac{\partial^2}{\partial x_i^2} - \frac{1}{2}A_{ik}\sigma_k'(F_k)\frac{\partial}{\partial x_i}$$

- Isomorphic (up to boundary) to Euclidean geometry:

$$d(\mathbf{F}_0, \mathbf{F}) = |\mathbf{x}_0 - \mathbf{x}|$$

- Geodesics known in closed form
- CEV case: $\sigma_i(F_i) = \sigma_i F_i^{\beta_i}$, zeroth and first order heat kernel coefficients given explicitly

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Outline

- Optimization problem for $\mathbf{F}^*$ is non-linear with a linear constraint
- With $q_i := \int_{F_{0,i}}^{F_i} \frac{du}{\sigma_i(u)}$, it is a quadratic optimization problem with non-linear constraint
- Fast convergence of Newton iteration for suitable initial guess
- Given $\mathbf{F}^*$, $C(\mathbf{F}_0, \mathbf{F}^*)$ is a line integral along the geodesic; this integral can be calculated in closed form in the CEV model.
- Formulas can be evaluated in less than 2 seconds for $n = 100$

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- CEV model framework
- For CEV, the formulas are fully explicit apart from the minimizing configuration $\mathbf{F}^*$
- We observe very fast convergence of the iteration, but the initial guess is crucial.
- Reference values obtained using:
  - *Ninomiya Victoir* discretization
  - Quasi Monte Carlo based on Sobol numbers, Monte Carlo for very high dimensions ($n \approx 100$)
  - Variance (dimension) reduction using *Mean value Monte Carlo* based on one-dimensional Black-Scholes prices

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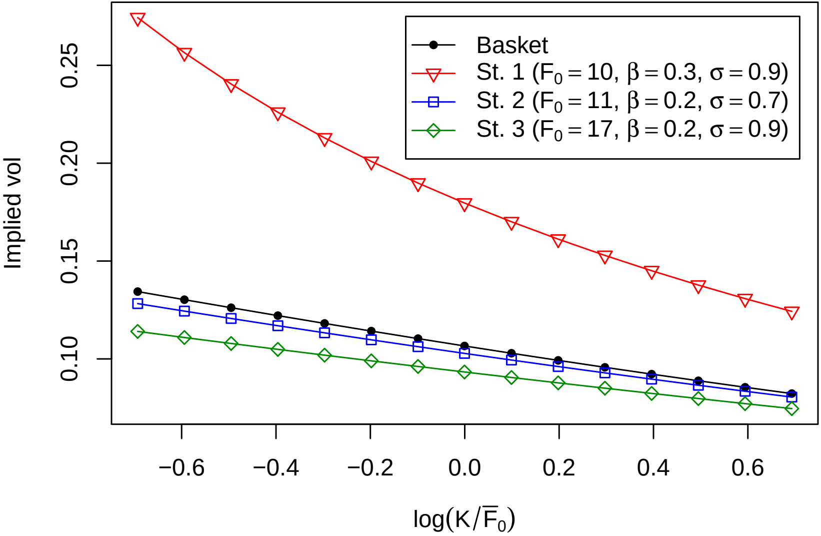
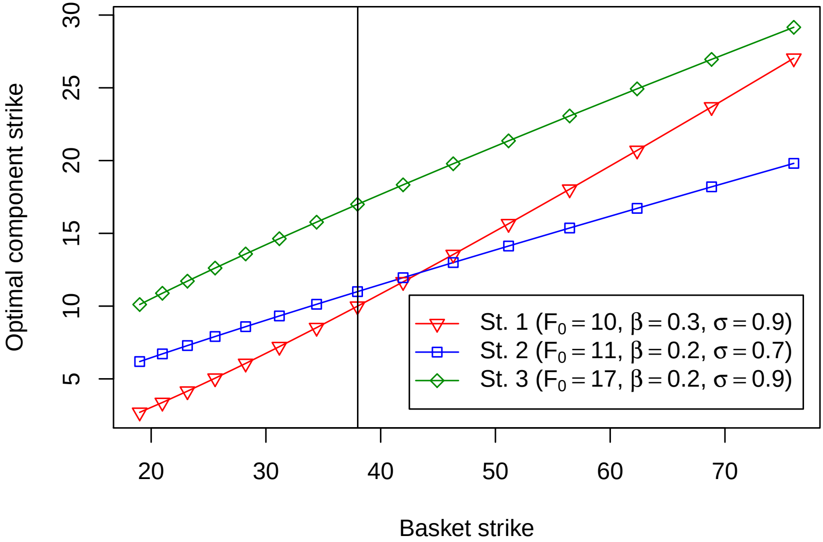

Basket
St. 1 ($F_0=10$, $\beta=0.3$, $\sigma=0.9$)
St. 2 ($F_0=11$, $\beta=0.2$, $\sigma=0.7$)
St. 3 ($F_0=17$, $\beta=0.2$, $\sigma=0.9$)
Implied vol
0.10
0.15
0.20
0.25
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
$\log(K/\overline{F}_0)$

CEV index implied vol – three-dimensional visualization



Normalized errors

- Approximation error supposed to depend on "dimension-free" time to maturity $\sigma^2 T$
- Use $\overline{\sigma} := \sigma_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0) / \left(\sum_{i=1}^{n} w_i F_{0,i}\right)$ as proxy in local vol framework
- Normalized error: $\frac{\text{Rel. error}}{\overline{\sigma}^2 T}$

T	Dim. 5	Dim. 10	Dim. 15	Dim. 100
0.5	0.1555	−0.0293	0.3085	−0.0143
1	0.1481	−0.0261	0.3162	−0.0105
2	0.1429	−0.0218	0.3222	−0.0075
5	0.1376	−0.0129	0.3252	
10	0.1328	−0.0035	0.3198	
$\overline{\sigma}$	0.1704	0.3187	0.1073	0.2964

Table: Normalized relative error of the zero-order asymptotic prices.

- Approximation error supposed to depend on "dimension-free" time to maturity $\sigma^2 T$
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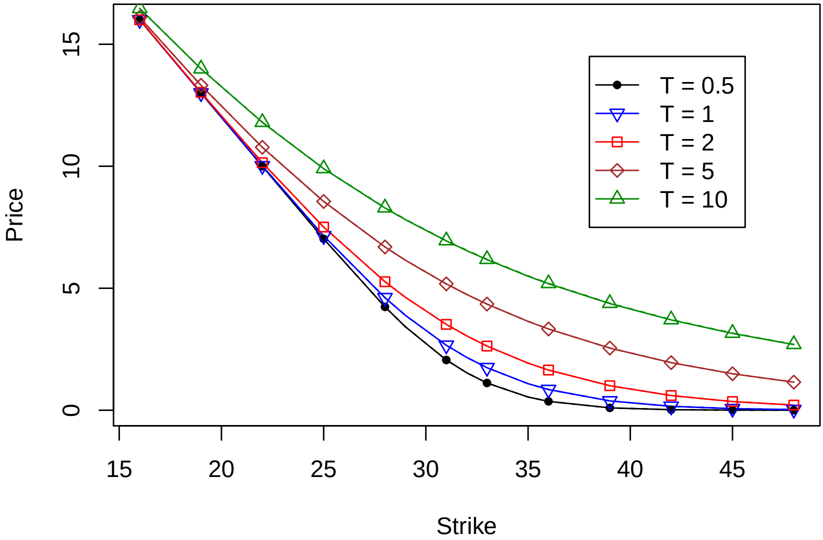
- Normalized error: $\frac{\text{Rel. error}}{\overline{\sigma}^2 T}$

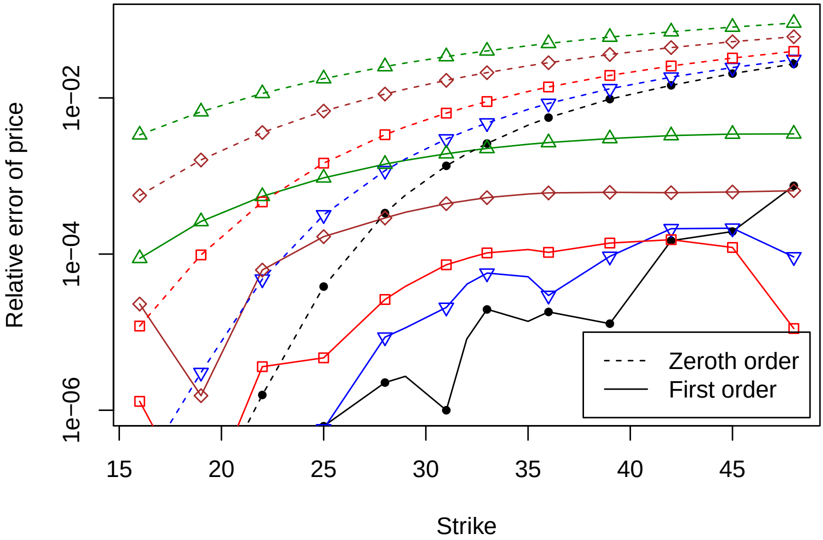
T	Dim. 5	Dim. 10	Dim. 15	Dim. 100
0.5	-4.02×10^{-4}	1.76×10^{-4}	8.76×10^{-3}	5.06×10^{-5}
1	-9.47×10^{-4}	3.58×10^{-3}	1.53×10^{-3}	2.08×10^{-3}
2	-1.63×10^{-3}	8.09×10^{-3}	-3.92×10^{-3}	3.89×10^{-3}
5	-3.41×10^{-3}	1.71×10^{-2}	-1.33×10^{-2}	
10	-7.15×10^{-3}	2.67×10^{-2}	-2.82×10^{-2}	
$\overline{\sigma}$	0.1704	0.3187	0.1073	0.2964

Table: Normalized error of the first order asymptotic prices.

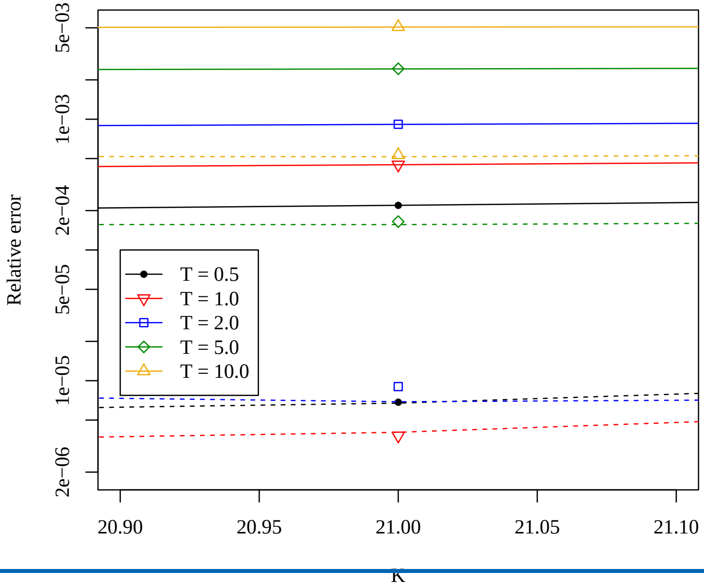
First order prices



Relative errors



Relative error ATM



Delta

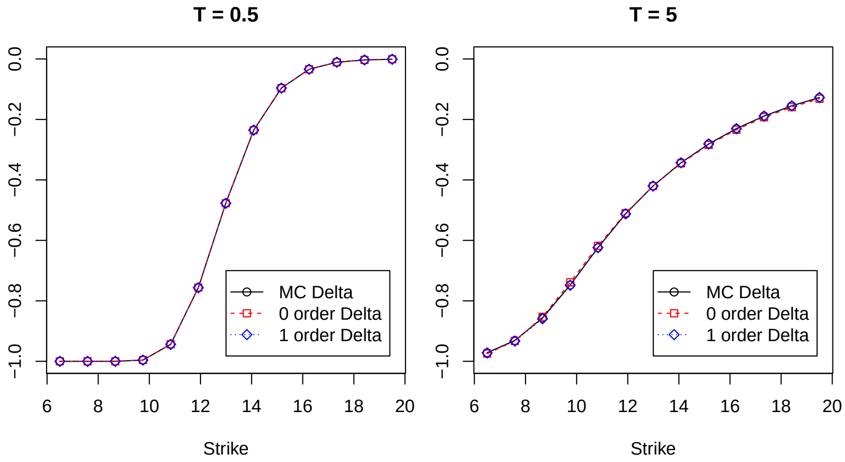
$$\mathbf{F}_0 = \begin{pmatrix} 13 \\ 9 \\ 9 \end{pmatrix}, \ \boldsymbol{\xi} = \begin{pmatrix} 0.1 \\ 0.7 \\ 0.6 \end{pmatrix}, \ \boldsymbol{\beta} = \begin{pmatrix} 0.3 \\ 0.7 \\ 0.5 \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1.0000 & 0.9142 & 0.7706 \\ 0.9142 & 1.0000 & 0.8429 \\ 0.7706 & 0.8429 & 1.0000 \end{pmatrix}.$$

- Objective: Compute the sensitivity (delta) w.r.t.$F_{0,3}$.
- Note that the option payoff is

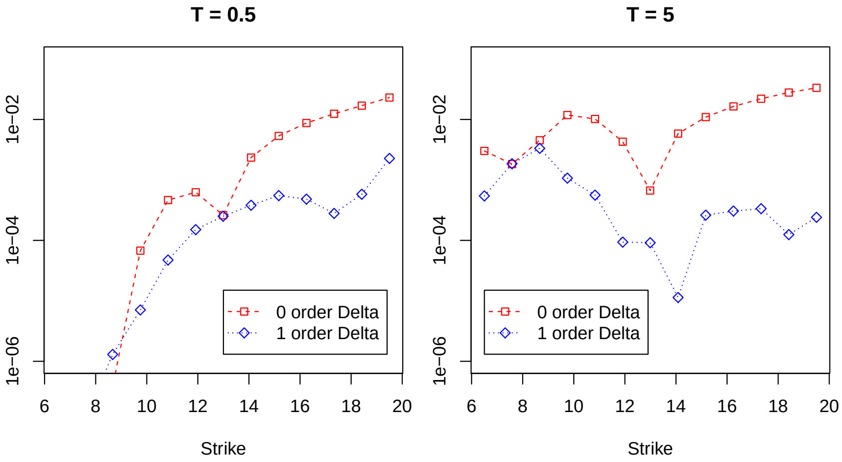
$$P(\mathbf{F}) = (F_1 + F_2 - F_3 - K)^+$$

Delta



T = 0.5
T = 5
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
6
8
10
12
14
16
18
20
Strike
MC Delta
0 order Delta
1 order Delta

Relative error of delta



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